18.100A PSET 1 SOLUTIONS

DAVID CORWIN

Problem 1

Let $a_n = (-1)^n$. First, we note that for all n, we either have $a_n = 1$ or $a_n = -1$. In either case, we have $-1 \leq a_n \leq 1$. Therefore, $\{a_n\}$ is bounded below by -1 and above by 1, i.e., it is bounded.

For the next part, we present two possible solutions:

Solution 1. Next, let us suppose that the sequence has a limit, call it L. Then for some $N > 0$, we have $|a_n - L| < 1/2$ whenever $n > N$.

But for any such n, we can always find an even n such that $n > N$. Then for such an n, we have $a_n = 1$, so $|L - 1| < 1/2$, hence $L > 1/2$. Similarly, we can always find odd n such that $n > N$, so $a_n = -1$ for such an n, and so $|(-1) - L| < 1/2$, so $L < -1/2$. But this contradicts $L > 1/2$, so a limit cannot exist.

Solution 2. We consider the subsequences $\{a_{2n}\}$ and $\{a_{2n+1}\}$. By the Subsequence Theorem, if the sequence $\{a_n\}$ has a limit, then any subsequence has the same limit. But the first subsequence has limit 1, and the second has limit -1, a contradiction, so the original sequence cannot have a limit.

Comments.

- Some people's explanations were overly complicated
- The key is to choose epsilon less than 1 (or realize that any such epsilon works).
- A lot of people bounded it by ± 2 or even ± 10. That might be good for intuition, but you only need ± 1.
- Some people said "choose some n even" or "choose some n odd" and that the limit is then 1 (or -1). But a limit isn't about a single

value (“some” n), rather it’s about what happens as n (say, even) gets larger and larger.
- An important point people weren’t making explicit: you need to note that there are *arbitrarily large* even and odd n.

Problem 2

Letting $a_n = \dfrac{n-1}{3^n}$, we have

$$\begin{aligned} a_{n+1} - a_n &= \frac{n}{3^{n+1}} - \frac{n-1}{3^n} \\ &= \frac{n}{3^{n+1}} - \frac{3n-3}{3^{n+1}} \\ &= \frac{n-(3n-3)}{3^{n+1}} \\ &= \frac{-2n+3}{3^{n+1}}. \end{aligned}$$

For $n \geq 2$ (in fact any $n > 3/2$), we have $-2n+3 < 0$, and, noting that $3^{n+1} > 0$, this implies that $a_{n+1} - a_n < 0$. But this implies that $a_{n+1} < a_n$, which, by definition, says that the sequence is decreasing.

Comments.

- Some people are starting with the conclusion and then getting to an inequality that’s true. If you do that, you need to explain that the steps are reversible!

Problem 3

To show that the sequence is bounded above, note that for $n \geq 1$, we have $a_n = -a_{n-1}^2$, which is ≤ 0. Noting that $a_0 < 0$, we have $a_n \leq 0$ for all $n \geq 0$, so the sequence is bounded above by 0.

To show that the sequence is increasing, we need to know that $a_n \geq -1$ for all n. Such a property for a_n depends on the same property for a_{n-1}, so we need to use induction (notice how, in the previous paragraph, $-a_{n-1}$ is ≥ 0 regardless of the value of a_{n-1}, so we do not need induction).

For $n = 0$, we have $a_0 \geq -1$. For some $n \geq 0$, suppose $a_n \geq -1$. We also know that $a_n < 0$ (by the first paragraph), so $a_n^2 = (-a_n)^2$ is between 0 and 1, meaning that $a_{n+1} = -a_n^2$ is ≥ -1, as desired. By induction, we have $a_n \geq -1$ for all n.

We have now shown that $-1 \leq a_n < 0$ for all $n \geq 0$. It follows that $-a_n$ is positive, so we may multiple both sides of the inequality $-1 \leq a_n$ to get the inequality $a_n \leq -a_n^2$. But this says that $a_n \leq a_{n+1}$, which implies that the sequence is increasing.

Comments.

- It's important to understand where induction is needed and where it isn't needed. A lot of people used induction on the wrong part.
- As an alternative proof, one can show $a_n = -a_0^{2^n}$ for all n and proceed directly (i.e., without even using induction). (Though technically, proving that formula involves induction, albeit a very intuitive example of induction.)

Problem 4

We first note that $a_n > 0$ for all n, as the numerator and denominator are clearly positive. In particular, this implies that the sequence is bounded below.

Next, we note that

$$\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{\frac{2^{2n+2}((n+1)!)^2}{(2n+3)!}}{\frac{2^{2n}(n!)^2}{(2n+1)!}} \\
&= \frac{\frac{2^{2n+2}((n+1)!)^2}{2^{2n}(n!)^2}}{\frac{(2n+3)!}{(2n+1)!}} \\
&= \frac{4(n+1)^2}{(2n+3)(2n+2)} \\
&= \frac{2n+2}{2n+3} \\
&< 1.
\end{aligned}$$

As everything is positive, this implies that $a_{n+1} < a_n$, i.e., the sequence is decreasing. But a bounded below decreasing sequence always has a limit by the Completeness Theorem, so it has a limit.

Comments.

- One can also show that the limit is 0 by noting that $a_0 = 1$, so $a_n = \prod_{k=1}^{n} \frac{2k}{2k+1} = \prod_{k=1}^{n} \frac{1}{1+\frac{1}{2k}} < \frac{1}{\sum_{k=1}^{n} \frac{1}{2k}}$, but the bottom diverges to ∞, so the limit approaches 0.

Problem 5

Solution 1. We let $H_n = \sum_{k=1}^{n} \frac{1}{k}$ denote the Harmonic series. We note that $a_n = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1} > \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n-2} = \frac{1}{2}H_{n-1}$. But we know that the Harmonic series grows arbitrarily large, hence so does a_n. (More precisely, if $a_n < B$ for some B and all n, the above would imply that $H_{n-1} < 2B$ for all n, contradicting the fact that H_n is unbounded.)

Solution 2. We note that a_n is an upper Riemann sum for the integral $\int_1^{2n+1} \frac{1}{2x-1}dx$. The antiderivative of the integrand is $\frac{\log(2x-1)}{2}$, so the integral evaluates to $\frac{\log(2(n+1)-1)}{2} - \frac{\log(2-1)}{2} = \frac{1}{2}\log(2n+1)$. But the log function is unbounded so this approaches $+\infty$, hence so does a_n.

Comments.

- Be careful, a_n is the sum of n terms, so you need to compare it to an integral from 1 to $n+1$, not from 1 to n

Problem 6

There is a counterexample. We choose any sequences a_n and b_n such that each is increasing and always negative. For example, let $a_n = b_n = -\frac{1}{n}$. Then $|a_n|$ is decreasing, hence so is a_n^2, and the same is true for b_n^2, so

$a_n^2 + b_n^2$ is also decreasing. (In that example, we have $a_n^2 + b_n^2 = \frac{2}{n^2}$, which is decreasing.)

Comments.

- Technically, you can find something where $a_n^2 + b_n^2$ increases for sufficiently large n, but fails in general due to behavior for small values of n. But it's more interesting to note that there's a counterexample even if you write "increasing for sufficiently large n" instead of "increasing."
- Note that the word 'counterexample' has the word 'example' in it, and although I didn't take off points for this, it's good to include an example.

Problem 7

We first prove that for all n, we have $0 \leq a_n < \frac{2}{\sqrt{3}}$. We do this by induction. For $n = 0$, this is automatically true. Now assume that $0 \leq a_n < \frac{2}{\sqrt{3}}$ for some n.

Then a_{n+1} is a positive square root, so it is clearly ≥ 0. We also know $a_n^2 < 4/3$, so $a_{n+1}^2 = 1 + a_n^2/4 < 1 + (4/3)/4 = 1 + 1/3 = 4/3$, which implies that $a_{n+1} < \frac{2}{\sqrt{3}}$.

Comments.

- When proving this by induction, it's important to be clear about *what statement* you're trying to prove by induction. This was confusing in a lot of the problem sets.
- As an alternative proof, one can actually find a closed form for a_n. More specifically, one has $a_n^2 = \frac{a_0^2}{4^n} + \sum_{k=1}^{n-1} \left(\frac{1}{4}\right)^{k-1} = \frac{a_0^2}{4^n} + \frac{4 - \left(\frac{1}{4}\right)^{n-1}}{3}$, so the limit of a_n is $\frac{2}{\sqrt{3}}$.

Problem 8

(b). We have

$$\begin{aligned}\left|\left(\frac{2n-1}{n+2}\right)-2\right| &= \left|\frac{-5}{n+2}\right| \\ &= \frac{5}{n+2} \\ &< \frac{5}{n}\end{aligned}$$

For any $\epsilon > 0$, this is less than ϵ for $n > \frac{5}{\epsilon}$. This implies that $\lim_{n\to\infty} \frac{2n-1}{n+2} = 2$.

(c). As $n > 0$, we have

$$\begin{aligned}\left|\frac{n}{n^2+3n+1}\right| &= \frac{n}{n^2+3n+1} \\ &< \frac{n}{n^2} \\ &= \frac{1}{n}\end{aligned}$$

For any $\epsilon > 0$, this is less than ϵ for $n > \frac{1}{\epsilon}$, so $\lim_{n\to\infty} \frac{n}{n^2+3n+1} = 0$

(e). We note that $(\sqrt{n^2+2}-n)(\sqrt{n^2+2}+n) = n^2+2-n^2 = 2$, so $\sqrt{n^2+2}-n = \frac{2}{\sqrt{n^2+2}+n}$. For $n > 0$, we have $\sqrt{n^2+2}$ is well defined and ≥ 0, so

$$\begin{aligned}|\sqrt{n^2+2}-n| &= \left|\frac{2}{\sqrt{n^2+2}+n}\right| \\ &= \frac{2}{\sqrt{n^2+2}+n} \\ &< \frac{2}{n}\end{aligned}$$

For any $\epsilon > 0$, this is less than ϵ for $n > \frac{2}{\epsilon}$. This implies that $\lim_{n\to\infty} \sqrt{n^2+2} - n = 0$.

Problem 9

(a). We first note that

$$\begin{aligned} a_n - a_{n-1} &= \left(\frac{1}{n+1} + \cdots + \frac{1}{2n}\right) - \left(\frac{1}{n} + \cdots + \frac{1}{2n-2}\right) \\ &= \frac{1}{2n} + \frac{1}{2n-1} - \frac{1}{n} \\ &= \frac{1}{2n-1} - \frac{1}{2n} \\ &= \frac{2n}{2n(2n-1)} - \frac{2n-1}{2n(2n-1)} \\ &= \frac{1}{2n(2n-1)} \\ &> 0 \end{aligned}$$

In particular, we find that $\{a_n\}$ is increasing for $n \geq 1$.

Furthermore, we note that

$$a_n = \sum_{k=n+1}^{2n} \frac{1}{k} < \sum_{k=n+1}^{2n} \frac{1}{n+1} = \frac{n}{n+1} < 1,$$

so a_n is bounded above. By the Completeness Theorem, it follows that a_n has a limit.

(b). In the K-ϵ principle, K must be a constant. But this proposed "proof" is taking $K = n$, which is not constant.

Problem 10

We will actually do both cases at once. Let M be a positive integer greater than $2|r|$. We note that for $n \geq M$, we have $\frac{|r|}{n} < \frac{1}{2}$. Therefore, for $n > M$, we have

$$\begin{aligned} |a_n| &= \frac{|r|^n}{\prod_{k=1}^{n} k} \\ &= |a_M|\frac{|r|^{n-M}}{\prod_{k=M+1}^{n} k} \\ &= |a_M|\prod_{k=M+1}^{n}\frac{|r|}{k} \\ &< |a_M|\prod_{k=M+1}^{n}\frac{1}{2} \\ &= |a_M|\left(\frac{1}{2}\right)^{n-M} \end{aligned}$$

As $\lim_{n\to\infty}\left(\frac{1}{2}\right)^{n-M} = 0$, we know that for any $\epsilon > 0$, that $|a_n| = |a_M|\left(\frac{1}{2}\right)^{n-M} < |a_M|\epsilon$ for $n >> 1$. By the K-ϵ principle, it follows that $\lim_{n\to\infty} a_n = 0$.